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Symmetric flows and Darcy's law in curved spaces

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Abstract

We consider the problem of existence of certain symmetrical solutions of the Stokes equation on a three-dimensional manifold M with a general metric possessing symmetry. These solutions correspond to unidirectional flows. We have been able to determine necessary and sufficient conditions for their existence. Symmetric unidirectional flows are fundamental for deducing the so-called Darcy's law, which is the law governing fluid flow in a Hele–Shaw cell embedded in the environment M. Our main interest is to depart from the usual, flat background environment and consider the possibility of an environment of arbitrary constant curvature K in which a cell is embedded. We generalize Darcy's law for particular models of such spaces obtained from $\mathbb{R}^3$ with a conformal metric. We employ the calculus of differential forms for a simpler and more elegant approach to the problems discussed.

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1. Introduction

Pattern formation is a very exciting and fast growing area in physics and related sciences [1–5]. The Saffman–Taylor problem [6] is one of the most studied among the systems presenting the formation and evolution of patterned structures. It studies the hydrodynamic instabilities at the interface separating two immiscible fluids confined between two parallel flat plates, the Hele–Shaw cell. In such a configuration, when a low-viscosity fluid displaces a higher viscosity fluid, the interface becomes unstable, deforms and forms fingers [7–10] (see figure 1). The key point in the study of such patterns is Darcy's law, the two-dimensional reduction of the Navier–Stokes equation incorporating the boundary conditions (no-slip) and the mass conservation law (continuity equation). Curiously, Darcy's law, which describes the dynamic behaviour for flow in *flat* Hele–Shaw cells, is actually the very same equation

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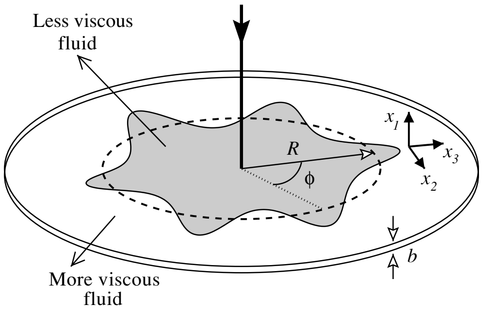

Figure 1. Schematic configuration of the radial flow in a *flat* Hele–Shaw cell. The less viscous fluid (dark fluid) is injected into the Hele–Shaw cell, previously filled with the more viscous fluid. The dashed line represents the initial unperturbed circular interface (radius R) and the solid undulated curve depicts the deformed fluid–fluid interface at later times. The thickness of the cell is denoted by b. Using the cylindrical coordinate system $(x_1, x_2, x_3) = (z, \phi, \rho)$, we assume that the fluid motion occurs along the radial direction ($x_3 = \rho$), while the transversal direction x_1 is simultaneously perpendicular to both the upper ($z = b$) and the lower ($z = 0$) plates.

used in the description of flow in porous media [6–9]. This happens despite the fact that the latter is indeed a highly *non-flat* environment, characterized by voids and curved internal surfaces. The question arises as to whether there exists a deeper connection between the fingering phenomena and the geometric and topological features of the substrate on which the flow takes place.

Considering the wealth of interesting phenomena already found in the flat version of the Saffman–Taylor problem, there is an obvious scientific interest in the study of fingering in *curved* Hele–Shaw cells. On a more practical level, the curved problem may have applications in a number of industrial and manufacturing processes involving the filling of a thin cavity between two walls of a given shape with fluid. These processes range through pressure moulding of molten metals and polymer materials [11], and formation of coating defects in drying paint thin films [12]. Another potentially promising system refers to recent microfluidic applications, where the fluid is confined to flow in tiny microchannels of various shapes [13]. Generalizations of the standard case of flat Hele–Shaw flows are also useful and of interest in the treatment of foams in curved surfaces [14–16].

Motivated by such scientific and practical aspects, researchers started to investigate the impact of cell geometry and topology on the Saffman–Taylor problem [17–20]. Spherical [17, 18], cylindrical [19] and conical [20] geometries have been studied in this context, yielding substantial information connecting relevant fingering mechanisms (finger competition and finger tip splitting) to the cell's geometric and topological features. A schematic representation of some curved Hele–Shaw cells is depicted in figure 2: spherical (top), cylindrical (centre), and conical (bottom). In spherical cells [17], it has been shown that the cell's Gaussian curvature regulates finger tip-splitting behaviour. In contrast, the relevant control parameter in cylindrical cells [19] is the mean curvature, which determines the strength of the competition among the fingers. Topology rather than geometry seems to be the key factor in determining the shape of the fluid–fluid interface in conical cells [20], where the emerging patterns are significantly sensitive to variations in the conical cell's opening angle. Here, our aim is to provide the means of further generalizations of the Saffman–Taylor problem involving other geometries, opening the possibility of investigating how aspects related to the curved nature of the Hele–Shaw cell may determine the morphology and evolution of the fingering structures.

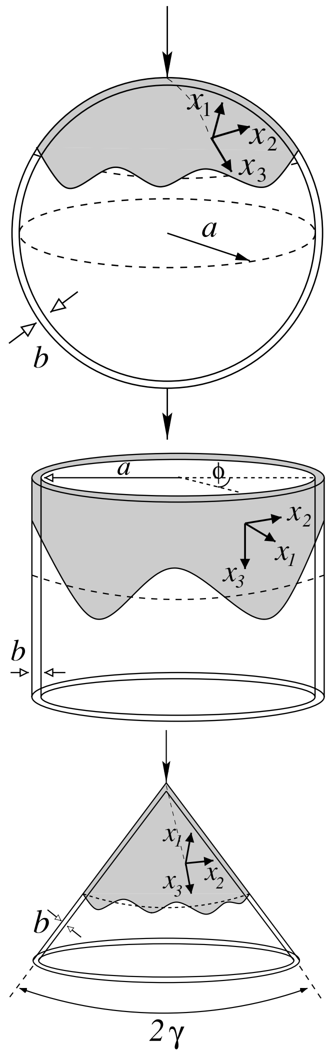

Figure 2. Illustrative examples of *curved* Hele–Shaw cells. Similarly to figure 1, here the dark fluid is less viscous, the solid undulated curve represents the deformed fluid–fluid interface and b denotes the cell thickness. In the spherical cell (top) of inner radius a we adopt the coordinates $(x_1, x_2, x_3) = (r, \phi, \theta)$. For the cylindrical cell (centre) of inner radius a, we have that $(x_1, x_2, x_3) = (\rho, \phi, z)$. In the conical cell (bottom) of opening angle 2γ, we use $(x_1, x_2, x_3) = (\xi, \phi, \rho)$. Note that throughout this work (see section 2 for details) we assume that the fluid motion is in the direction x_3, while the transversal direction to the plates is defined as x_1.

In this work, we present a systematic and rigorous derivation of Darcy's law by averaging the Stokes and continuity equations for fluid flow in curved spaces, and find the conditions under which a unidirectional symmetric flow exists for more general *non-flat* Hele–Shaw cells. We allow the ordinary three-dimensional space $\mathbb{R}^3$ to acquire a symmetrical Riemannian metric [21], then look for the conditions wherein Navier–Stokes equation in this space becomes separable. An important case where this occurs is that of a separable metric. For this case, we demonstrate that symmetric unidirectional flows are possible. Another important case is that of $\mathbb{R}^3$ endowed with a conformal metric. For this case, we derive Darcy's law even in the situations where symmetric unidirectional flows do not exist. The case of a pseudo-Riemannian metric (Minkowski) is also approached for it supports surfaces of constant negative curvature (pseudo-spheres or Lobachevsky planes), which have been partially studied in [17]. The calculus of differential forms [22] is used throughout the paper for a simpler and more elegant way of presenting the problem.

Suppose M is a smooth orientable manifold of dimension 3 endowed with a metric locally given by

$$\mathrm{d}s^2 = E_1^2\,\mathrm{d}x_1^2 + E_2^2\,\mathrm{d}x_2^2 + E_3^2\,\mathrm{d}x_3^2, \tag{1}$$

where E_i are the smooth functions of the coordinates $x_i, i = 1, 2, 3$. Let $(,)$ be the inner product induced by this metric.

The motion of a fluid in M is described by a vector (velocity) field $\vec{V} : M \to TM$, where TM denotes the tangent bundle to M. There exists a canonical correspondence between TM and the cotangent bundle $TM^* = \bigwedge^1(TM^*)$, which is defined using the metric (1): to each tangent vector $\vec{V} \in TM$ there corresponds a unique differential 1-form $\omega_{\vec{V}} \in \bigwedge^1(TM^*)$ such that $\omega_{\vec{V}}(\vec{W}) = (\vec{V}, \vec{W})$. Given an orthonormal basis $\beta = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of TM, we define the corresponding basis of $\bigwedge^1(TM^*)$ as $\beta^* = \left\{\omega_{\vec{e}_1}, \omega_{\vec{e}_2}, \omega_{\vec{e}_3}\right\}$. The basis β^* is orthonormal in the inner product $\langle , \rangle$ induced by the dual metric

$$(\mathrm{d}s^*)^2 = \left(1/E_1^2\right)\mathrm{d}x_1^2 + \left(1/E_2^2\right)\mathrm{d}x_2^2 + \left(1/E_3^2\right)\mathrm{d}x_3^2. \tag{2}$$

Fluid motion is governed by the Navier–Stokes equation [23, 24]

$$\varrho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}, \nabla)\vec{V}\right] = -\mathrm{grad}(p) + \eta\Delta\vec{V},$$

where ϱ denotes the fluid density, η represents the fluid viscosity and p is the hydrodynamic pressure. If the flow is incompressible, $\vec{V}$ must also satisfy the equation of continuity

$$\mathrm{div}\,\vec{V} = 0.$$

In many applications, such as the study of Hele–Shaw flows, one assumes a steady flow and neglects the so-called inertial terms on the left-hand side of the Navier–Stokes equation. Under these hypotheses, Navier–Stokes equation reduces to Stokes equation. Using the correspondence between vector fields and differential forms, Stokes equation and the equation of continuity translate, respectively, into

$$-\omega_{\mathrm{grad}(p)} + \eta\Delta\omega_{\vec{V}} = 0, \tag{3}$$

$$\delta\omega_{\vec{V}} = 0, \tag{4}$$

where Δ is the Laplace operator

$$-(\mathrm{d}\delta + \delta\,\mathrm{d}).$$

We recall that the operators d and δ are the exterior differential and codifferential, respectively. The codifferential is an operator from $\bigwedge^k(TM^*)$ to $\bigwedge^{k-1}(TM^*)$ defined by

$$\delta = (-1)^k * \mathrm{d}*,$$

where $* : \bigwedge^k(TM^*) \to \bigwedge^{3-k}(TM^*)$ is the Hodge star operator.

Darcy's law is obtained by averaging $\vec{V}$ in the normal direction with respect to a given two-dimensional smooth submanifold N of M. It provides a reasonable description of the fluid motion between two non-intersecting neighbouring copies of N. Generally, such pair of submanifolds is said to form a Hele–Shaw cell. Of particular interest, due to their simplicity, are the cells formed by level sets $\{x_i = \text{constant}\}$ in a local chart of M. All examples of Hele–Shaw cells studied so far (planar, cylindrical, conical, spherical) are formed by such submanifolds. One procedure for obtaining Darcy's law consists of considering a one-parameter family of velocity fields which corresponds to what we called symmetric unidirectional flows. When such family exists and its profile function (denoted in this paper by g) is non-constant, a simple method provides a quick deduction of Darcy's law. The existence of symmetric unidirectional flows is, therefore, a very important issue. One of our main goals is to attempt to overcome the non-existence of such flows in a perturbative way.

The paper is organized as follows. In section 2, we look for solutions of Stokes equations for a symmetric unidirectional flow and analyse under which conditions the equation is separable. In section 3, we find the conditions for the existence of symmetric flows in curved space; and in section 4, we study the solutions of Stokes equation (3) and deduce Darcy's law for the following systems of the Hele–Shaw type:

(1) Two nearby pseudo-spheres in Minkowski's 3-space.
(2) Two parallel planes in $\mathbb{R}^3$ with a conformal metric.

Section 5 summarizes our main results and conclusions.

2. Solution of Stokes equation for a symmetric unidirectional flow

We study the solutions of Stokes equation in M under the following assumptions:

(A1) The level sets $S_a = \{x_1 = a\}$ and $S_{a+b} = \{x_1 = a + b\}$ are two smooth (non-intersecting) surfaces. The vector field $\vec{e}_1$ is normal to both S_a and S_{a+b}.
(A2) The coefficients of the metric (1) of M do not depend on x_2.
(A3) Fluid motion is in the direction of x_3 and the velocity field does not depend on x_2. We refer to such motion as a *symmetric unidirectional flow*.

A velocity field satisfying assumption (A3) has the form

$$\vec{V} = V_3(x_1, x_3)\vec{e}_3.$$

Using the correspondence with 1-forms, we obtain

$$\omega_{\vec{V}} = V_3\omega_{\vec{e}_3} = (V_3 E_3)\,\mathrm{d}x_3.$$

The equation of continuity then writes

$$\begin{aligned}\delta\omega_{\vec{V}} &= -*\,\mathrm{d}*(V_3E_3\,\mathrm{d}x_3) = -*\,\mathrm{d}(V_3E_1\,\mathrm{d}x_1 \wedge E_2\,\mathrm{d}x_2)\\ &= -*\left(\frac{\partial(V_3E_1E_2)}{\partial x_3}\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3\right) = 0.\end{aligned}$$

Thus, we have $\frac{\partial(E_1E_2V_3)}{\partial x_3} = 0$, which implies that

$$E_1E_2V_3 = g(x_1) \tag{5}$$

for some function g which remains to be determined.

Using the equation of continuity, the Laplacian of $\omega_{\vec{V}}$ reduces to $-\delta\,\mathrm{d}\omega_{\vec{V}}$, i.e.,

$$\begin{aligned}-*\,\mathrm{d}*\,\mathrm{d}(V_3E_3\,\mathrm{d}x_3) &= -*\,\mathrm{d}*\left(\frac{\partial(V_3E_3)}{\partial x_1}\,\mathrm{d}x_1 \wedge \mathrm{d}x_3\right)\\ &= -*\,\mathrm{d}\left(-\frac{E_2}{E_1E_3}\frac{\partial(V_3E_3)}{\partial x_1}\,\mathrm{d}x_2\right)\end{aligned}$$

which equals

$$*\left(\frac{\partial}{\partial x_1}\left(\frac{E_2}{E_1E_3}\frac{\partial(V_3E_3)}{\partial x_1}\right)\mathrm{d}x_1\wedge\mathrm{d}x_2-\frac{\partial}{\partial x_3}\left(\frac{E_2}{E_1E_3}\frac{\partial(V_3E_3)}{\partial x_1}\right)\mathrm{d}x_2\wedge\mathrm{d}x_3\right)$$
$$=-\frac{E_1}{E_2E_3}\frac{\partial}{\partial x_3}\left(\frac{E_2}{E_1E_3}\frac{\partial(V_3E_3)}{\partial x_1}\right)\mathrm{d}x_1+\frac{E_3}{E_1E_2}\frac{\partial}{\partial x_1}\left(\frac{E_2}{E_1E_3}\frac{\partial(V_3E_3)}{\partial x_1}\right)\mathrm{d}x_3.$$

Recall that, by definition of the gradient,

$$\omega_{\mathrm{grad}(p)}=\mathrm{d}p=\frac{\partial p}{\partial x_1}\mathrm{d}x_1+\frac{\partial p}{\partial x_3}\mathrm{d}x_3.$$

Thus, from the expression for the Laplacian of $\omega_{\vec{V}}$ deduced above, Stokes equation is equivalent to the following system of equations:

$$-\frac{\partial p}{\partial x_1}-\eta\frac{E_1}{E_2E_3}\frac{\partial}{\partial x_3}\left(\frac{E_2}{E_1E_3}\frac{\partial(V_3E_3)}{\partial x_1}\right)=0, \tag{6}$$

$$-\frac{\partial p}{\partial x_3}+\eta\frac{E_3}{E_1E_2}\frac{\partial}{\partial x_1}\left(\frac{E_2}{E_1E_3}\frac{\partial(V_3E_3)}{\partial x_1}\right)=0. \tag{7}$$

The unknown function $g(x_1)$ in (5) must be such that the system above is satisfied for some smooth function $p(x_1,x_3)$. If we apply the operator d to both sides of Stokes equation

$$-\mathrm{d}p+\eta\Delta\omega_{\vec{V}}=0$$

we obtain, since $\eta\neq 0$,

$$\mathrm{d}\Delta\omega_{\vec{V}}=0$$

because $\mathrm{d}^2=0$. Hence, the (local) existence of p satisfying Stokes equation is guaranteed if g solves the equation

$$\mathrm{d}\Delta\omega_{\vec{V}}=-\mathrm{d}\delta\,\mathrm{d}\left(\frac{gE_3}{E_1E_2}\right)=0,$$

or, in coordinates, the equation

$$\frac{\partial}{\partial x_3}\left(\frac{E_1}{E_2E_3}\frac{\partial}{\partial x_3}\left(\frac{E_2}{E_1E_3}\frac{\partial\left(\frac{gE_3}{E_1E_2}\right)}{\partial x_1}\right)\right)=\frac{\partial}{\partial x_1}\left(\frac{E_3}{E_1E_2}\frac{\partial}{\partial x_1}\left(\frac{E_2}{E_1E_3}\frac{\partial\left(\frac{gE_3}{E_1E_2}\right)}{\partial x_1}\right)\right) \tag{8}$$

which is a linear ordinary differential equation of third order whose coefficients are functions of the metric coefficients. Thus, the existence of non-constant solutions of equation (8) depends on the metric. From now on we will only consider non-constant solutions of (8).

2.1. *Separable Stokes equation*

A great simplification is achieved when the metric is such that Stokes equation is reduced to a single separable differential equation. It turns out that for some important examples Stokes equation reduces to a separable equation (7). This fact motivates the definition below.

Definition 2.1. *We say that Stokes equation is* separable *if the system of differential equations (6) and (7) reduces to a separable equation (7).*

The following definition will also be useful in our discussion.

Definition 2.2. *A function $f(x_1,x_3)$ is* separable *if it can be written as a product of a function of x_1 and a function of x_3.*

Let us suppose that

$$\frac{\partial}{\partial x_3}\left(\frac{E_2}{E_1E_3}\frac{\partial}{\partial x_1}\left(\frac{gE_3}{E_1E_2}\right)\right)=0, \tag{9}$$

or, equivalently

$$\frac{E_2}{E_1E_3}\frac{\partial}{\partial x_1}\left(\frac{gE_3}{E_1E_2}\right)=C(x_1), \tag{10}$$

for some function C.

Multiplying both sides of the above equation by $g(x_1)$ and denoting $\frac{g}{E_1}$ by G, we obtain

$$G\left(\frac{E_2}{E_3}\right)\frac{\partial}{\partial x_1}\left(G\frac{E_3}{E_2}\right)=Cg.$$

Dividing both sides by G^2, we get

$$\frac{\partial}{\partial x_1}\ln\left(\frac{GE_3}{E_2}\right)=\frac{Cg}{G^2}\quad\Leftrightarrow\quad\ln\left(\frac{GE_3}{E_2}\right)=\int\frac{Cg}{G^2}\,\mathrm{d}x_1+\tilde{C}(x_3),$$

which in terms of g is

$$\ln\left(\frac{gE_3}{E_1E_2}\right)=\int\frac{CE_1^2}{g}\,\mathrm{d}x_1+\tilde{C}\quad\Leftrightarrow\quad\frac{E_3}{E_1E_2}=\frac{\exp(\tilde{C})\exp\left(\int\frac{CE_1^2}{g}\,\mathrm{d}x_1\right)}{g}.$$

The last equation above shows that equation (10) holds as long as the ratio $\frac{E_3}{E_1E_2}$ is the product of a function of x_1, a function of x_3 and a function of x_1, x_3.

Proposition 2.1. *Assume that $E_1=E_1(x_1)$. Then, Stokes equation is separable if and only if the ratio $\frac{E_3}{E_2}$ is separable.*

Proof. If $E_1=E_1(x_1)$ and if Stokes equation is separable in the sense of definition 2.1, then from equation (9)

$$\frac{E_3}{E_1E_2}=\frac{\exp(\tilde{C})\exp\left(\int\frac{CE_1^2}{g}\,\mathrm{d}x_1\right)}{g}\Rightarrow\frac{E_3}{E_2}=H(x_1)\tilde{H}(x_3)$$

for some functions $H, \tilde{H}$.

On the other hand, if $\frac{E_3}{E_2}=H(x_1)\tilde{H}(x_3)$ then (9) holds and equation (7) becomes

$$-\frac{\partial p}{\partial x_3}+\eta H(x_1)\tilde{H}(x_3)\frac{\partial}{\partial x_1}\left(\frac{E_2}{E_1E_3}\frac{\partial(V_3E_3)}{\partial x_1}\right)=0.$$

After dividing by $\tilde{H}(x_3)$ and moving the second term to the right-hand side, we have

$$\frac{1}{\tilde{H}}\frac{\partial p}{\partial x_3}=\eta H\frac{\partial}{\partial x_1}\left(\frac{E_2}{E_1E_3}\frac{\partial}{\partial x_1}\left(\frac{gE_3}{E_1E_2}\right)\right)=\eta H\frac{\partial}{\partial x_1}\left(\frac{1}{E_1H}\frac{\partial}{\partial x_1}\left(\frac{gH}{E_1}\right)\right).$$

Clearly each side of the above equation depends on a single variable. □

We now discuss a few examples.

Example 1. Let $M=\mathbb{R}^3$ with the Euclidean metric in cylindrical coordinates $(x_1,x_2,x_3)=(\rho,\phi,z)$. We have

$$E_1=E_3=1,\qquad E_2=\rho.$$

From proposition 2.1, it follows that Stokes equation for a flow in the z-direction reduces to equation (7)

$$\frac{\partial p}{\partial z} = \frac{\eta}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\left(\frac{g}{\rho}\right)\right). \tag{11}$$

If, instead, we consider flows in the ρ-direction (radial flow) which are ϕ-symmetric, then Stokes equation is

$$\frac{1}{\rho}\frac{\partial p}{\partial \rho} = \eta g''. \tag{12}$$

Finally, for z-symmetric flows in the ϕ-direction, we have

$$\frac{\partial p}{\partial \phi} = \eta\rho\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho g)\right). \tag{13}$$

Example 2. Let $M = \mathbb{R}^3$ with the Euclidean metric in spherical coordinates $(x_1, x_2, x_3) = (r, \phi, \theta)$. In this case

$$E_1 = 1, \qquad E_2 = r\sin\theta, \qquad E_3 = r.$$

Proposition 2.1 implies that Stokes equation for polar flows (that is, along the θ-direction) reduces to

$$\sin\theta\frac{\partial p}{\partial \theta} = \eta g''. \tag{14}$$

Example 3. Our arguments so far apply to semi-Riemannian metrics just as well. Let $M = \mathbb{R}^3$ with the Minkowski metric

$$\mathrm{d}s_M^2 = -\mathrm{d}r^2 + r^2\sinh^2\tau\,\mathrm{d}\phi^2 + r^2\,\mathrm{d}\tau^2$$

in pseudo-spherical coordinates $(x_1, x_2, x_3) = (r, \phi, \tau)$. The analogous of proposition 2.1 for a semi-Riemannian metric implies that Stokes equation reduces to

$$\sinh\tau\frac{\partial p}{\partial \tau} = \eta g''. \tag{15}$$

2.2. *An example of non-separable Stokes equation: flow in $\mathbb{R}^3$ with a conformal metric*

Suppose now M is $\mathbb{R}^3$ endowed with the metric

$$\mathrm{d}s^2 = f^2(x_3)\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right).$$

It turns out that such choices of M and $\mathrm{d}s^2$ provide examples of 3-spaces of arbitrary constant curvature [21]. Indeed, let K be a non-negative real number. If $f(x_3) = 1\big/\left(Kx_3^2 + 1/4\right)$, a direct calculations reveals that M has curvature K. For $f(x_3) = 1/(1 + \sqrt{K}x_3)$, we have that $\tilde{M} = \{(x_1, x_2, x_3) \in M/x_3 > 0\}$, the open half-space, has curvature $-K$.

Stokes equations (6) and (7) are

$$\frac{\partial p}{\partial x_1} = \eta\frac{2g'f'}{f^4}, \tag{16}$$

$$\frac{\partial p}{\partial x_3} = \eta\frac{g''}{f^3}. \tag{17}$$

If we apply condition (8) for the existence of functions g, p satisfying the above equations, we obtain

$$\left(-2\frac{f'}{f^4}\right)'g' = \frac{g'''}{f^3},$$

which is a separable equation. The above differential equation has a non-trivial solution if and only if $f^3\big(\frac{f'}{f^4}\big)'$ is constant, i.e., the conformal factor must satisfy a differential equation of the form

$$f^3\left(\frac{f'}{f^4}\right)' = \tilde{k}_1 \quad \Rightarrow \quad -\frac{2}{3}f^3\left(\frac{1}{f^3}\right)'' = k_1,$$

By setting $y = \frac{1}{f^3}$, we obtain the equation $y'' = -\frac{3}{2}k_1 y$, whose solutions are well known for any values of the constant k_1. For each solution $y = y(x_3)$, there corresponds an $f = y^{-1/3}$ and a g which is a solution of

$$g''' + k_1 g' = 0. \tag{18}$$

Thus, a restriction on the metric needs to be imposed in order to have non-trivial solutions of Stokes equation. Unfortunately, this restriction applies to the choices of conformal factors which give constant curvature. In particular, we have that symmetric unidirectional flows are not possible in hyperbolic 3-space.

3. The existence of symmetric unidirectional flows

Let us address the fundamental question of the existence of solutions of equation (8).

Recall that the existence of unidirectional flows depends on the existence of a solution $g = g(x_1)$ of equation (8). As we mentioned earlier, equation (8) is a third-order linear ordinary differential equation. It can be put in the form

$$g''' - A(x_1, x_3)g'' - B(x_1, x_3)g' - C(x_1, x_3)g = 0.$$

We will show that a differential equation such as this one can only have a solution if the coefficients A, B, C do not depend on x_3. We will need the following:

Lemma 3.1. *If the linear differential equation*

$$g'' - A(x_1, x_3)g' - B(x_1, x_3)g = 0$$

has a solution, then the coefficients A and B do not depend on x_3.

Proof. A solution of the equation in the statement satisfies the linear system

$$Ag' + Bg = g'' \qquad A_3 g' + B_3 g = 0,$$

where the subscript indicates partial derivative with respect to x_3. Thus the 2×2 determinant

$$\begin{vmatrix} A & B \\ A_3 & B_3 \end{vmatrix}$$

is zero. But from $A_3 g' + B_3 g = 0$, we have that A_3 is equal to a function μ of x_1 times B_3. Hence, we must have $A = \mu B$. So,

$$g'' = B(\mu g' + g).$$

Therefore, $B = B(x_1)$ and $A = A(x_1)$, as we wanted to prove. □

Lemma 3.2. *If the linear differential equation*

$$g''' - A(x_1, x_3)g'' - B(x_1, x_3)g' - C(x_1, x_3)g = 0$$

has a solution, then the coefficients A, B and C do not depend on x_3.

Proof. Consider the linear system

$$Ag'' + Bg' + Cg = g''', \tag{19}$$

$$A_3 g'' + B_3 g' + C_3 g = 0. \tag{20}$$

From the previous lemma, equation (20) implies that there exist functions μ and ν of x_1 such that $B_3 = \mu A_3$ and $C_3 = \nu A_3$. Integrating these equations with respect to x_3, we obtain

$$B = \mu A + \gamma(x_1), \qquad C = \nu A + \varpi(x_1).$$

Substituting into equation (19), we obtain

$$A(g'' + \mu g' + \nu g) + (\gamma g' + \varpi g) = g'''.$$

Therefore, A is a function of x_1 only, and the same must hold for B and C. □

Lemma 3.2 provides necessary conditions for the existence of symmetrical unidirectional flows on a 3-manifold M with a metric $\mathrm{d}s^2$ whose coefficients are E_1, E_2 and E_3. For instance, the coefficient of g'' in equation (8) is

$$A(x_1, x_3) = -\frac{\partial}{\partial x_1} \ln\left(\frac{E_3}{E_1^3 E_2}\right).$$

Lemma 3.2 says that the right-hand side is a function of x_1. Hence, a quick calculation shows that $\frac{E_3}{E_1^3 E_2}$ must be separable. We have proved the

Proposition 3.1. *A necessary condition for the existence of a solution of the differential equation (8) is that $\frac{E_3}{E_1^3 E_2}$ is separable.*

Propositions 2.1 and 3.1 imply the main result of this section.

Theorem 3.1. *Suppose $E_1 = E_1(x_1)$. If Stokes equation has a solution then $\frac{E_3}{E_2}$ is separable. Conversely, if $\frac{E_3}{E_2}$ is separable, then Stokes equation in the metric with coefficients E_1, E_2 and E_3 is separable and hence it has a solution.*

Theorem 3.1 implies the impossibility of symmetric unidirectional flows in conical or toroidal geometries.

Example 4. Let M be an open region of $\mathbb{R}^3$ endowed with the Euclidean metric in conical coordinates $(x_1, x_2, x_3) = (\xi, \phi, \rho)$

$$\mathrm{d}s^2 = \mathrm{d}\xi^2 + (\xi\cos\gamma + \rho\sin\gamma)^2\,\mathrm{d}\phi^2 + \mathrm{d}\rho^2,$$

where $0 < \gamma < \frac{\pi}{2}$ is a constant (the half-opening angle of the conical cell). Since $E_1 = 1$ and

$$\frac{E_3}{E_2} = \frac{1}{\xi\cos\gamma + \rho\sin\gamma}$$

is not separable, we have that equation (8) has no solution.

Example 5. Let M be an open region of $\mathbb{R}^3$ endowed with the Euclidean metric in toroidal coordinates $(x_1, x_2, x_3) = (r, \phi, \theta)$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + (a + r\cos\theta)^2\,\mathrm{d}\phi^2 + r^2\,\mathrm{d}\theta^2,$$

where $a > 0$ is constant. Since $E_1 = 1$ and

$$\frac{E_3}{E_2} = \frac{r}{a + r\cos\theta}$$

is not separable, it follows that (8) has no solution.

If E_1 depends on x_3, it is possible to have solvable Stokes equations which are not separable, as we saw in subsection 2.2.

Remark 3.2. Theorem 3.1 imposes a serious restriction on an argument that has been used to deduce Darcy's law for Hele–Shaw systems in curved geometries. In the next section, we discuss a method which provides a perturbed form of Darcy's law which is valid for the example in subsection 2.2, somehow bypassing the obstacle imposed by the non-existence of a symmetric unidirectional flow.

4. Darcy's law

We now deduce Darcy's law for a separable and a non-separable example of Hele–Shaw systems.

4.1. Pseudo-spheres in Minkowski space

This example well illustrates the method of deduction of Darcy's law for separable systems.

Recall that $x_1 = r$, $x_2 = \phi$ and $x_3 = \tau$ are the pseudo-spherical coordinates of M defined in example 3.

Consider the pseudo-spheres $S_a = \{r = a\}$ and $S_{a+b} = \{r = a + b\}$ in M. A curve going from $P \in S_a$ to S_{a+b} in the r-direction is given by a path $\lambda : [0, 1] \to M$, where

$$r(t) = r(P) + tb, \qquad \phi(t) = \phi(P), \qquad \tau(t) = \tau(P).$$

We average the function $V_\tau(r, \tau)$ along the path λ.

In order to solve equation (15), we set both sides equal to a constant C. If we impose the non-slip boundary conditions $g(a) = g(a + b) = 0$, then we must have

$$g(r) = -\frac{C}{2\eta}(r - a)(a + b - r). \tag{21}$$

The average of V_τ along λ is

$$\overline{V}_\tau = \frac{\int_\lambda V_\tau \,\mathrm{d}s}{\int_\lambda \mathrm{d}s},$$

where ds is the element of arc-length of λ. We have that

$$\overline{V}_\tau = \frac{\int_0^1 V_\tau(\lambda(t))\mathrm{i}b \,\mathrm{d}t}{\int_0^1 \mathrm{i}b \,\mathrm{d}t} = \frac{1}{b \sinh\tau} \int_a^{a+b} \frac{g(r)}{r} \,\mathrm{d}r.$$

Using expression (21) for g, it follows that

$$\overline{V}_\tau = -\frac{1}{2b\eta}\left(\int_a^{a+b} \frac{(r - a)(a + b - r)}{r} \,\mathrm{d}r\right) \frac{\partial p}{\partial\tau}.$$

Therefore, Darcy's law for two pseudo-spheres S_a, S_{a+b} in Minkowski's 3-space is

$$\overline{V}_\tau = -\frac{b^2 \mathcal{F}\left(\frac{b}{a}\right)}{12\eta}(\mathrm{grad}\, p)_\tau, \tag{22}$$

where $\mathcal{F}\left(\frac{b}{a}\right) = F(1, 2; 4; -b/a)$ is a hypergeometric function. For comparison, see the appendix of [17].

4.2. Parallel planes in $\mathbb{R}^3$ with a conformal metric

We return to the example discussed in subsection (2.2).

Let us consider the planes $S_a = \{x_1 = a\}$ and $S_{a+b} = \{x_1 = a + b\}$. We define a curve $\lambda : [0, 1] \to \mathbb{R}^3$ by

$$x_1(t) = x_1(P) + tb, \qquad x_2(t) = x_2(P), \qquad x_3(t) = x_3(P),$$

where $P \in S_a$. In order to simplify our calculations, we will assume that $a = 0$ and $b > 0$.

The average of a function $h : \mathbb{R}^3 \to \mathbb{R}$ along λ is

$$\overline{h} = \frac{\int_\lambda h\,\mathrm{d}s}{\int_\lambda \mathrm{d}s} = \frac{\int_0^1 h(\lambda(t)) f(x_3(P))\,\mathrm{d}t}{\int_0^1 f(x_3(P)) b\,\mathrm{d}t} = \frac{1}{b}\int_a^{a+b} h(x_1, x_2(P), x_3(P))\,\mathrm{d}x_1.$$

We will regard $\bar{\ }$ as an averaging operator, which has the property of being linear with respect to functions of x_3. Besides, we also have

$$\overline{\frac{\partial h}{\partial x_3}} = \frac{\partial \overline{h}}{\partial x_3}. \tag{23}$$

If we apply $\bar{\ }$ to both sides of equations (5), (16) and (17), we obtain

$$\overline{V}_3 = \frac{\overline{g}}{f^2}, \tag{24}$$

$$\overline{\frac{\partial p}{\partial x_1}} = \left(\frac{2\eta f'}{f^4}\right)\overline{g'}, \tag{25}$$

$$\overline{\frac{\partial p}{\partial x_3}} = \left(\frac{\eta}{f^3}\right)\overline{g''}. \tag{26}$$

If we impose the no-slip boundary conditions $g(0) = g(b) = 0$, then $\overline{g'} = 0$ and hence $\overline{\frac{\partial p}{\partial x_1}} = 0$. So, from now on we will drop equation (25).

As we have seen in subsection 2.2, the function $g(x_1)$ must satisfy

$$g''' + k_1 g' = 0, \tag{27}$$

where $k_1 = -\frac{2}{3} f^3 \left(\frac{1}{f^3}\right)''$. Equation (27) is equivalent to

$$g'' + k_1 g = k_2$$

for some constant k_2. The solution of this equation satisfying the no-slip boundary conditions is

$$g(x_1) = \frac{k_2}{k_1}\left[1 + \left(\frac{\sinh\alpha(x_1 - b) - \sinh\alpha x_1}{\sinh\alpha b}\right)\right], \tag{28}$$

where $k_1 = -\alpha^2$. The averages of g and g'' are thus

$$\overline{g} = \frac{k_2}{k_1}\left[1 + 2\left(\frac{1 - \cosh\alpha b}{\alpha b \sinh\alpha b}\right)\right], \qquad \overline{g''} = \frac{-2k_2}{\alpha b}\left[\left(\frac{1 - \cosh\alpha b}{\sinh\alpha b}\right)\right].$$

Using these formulae, we obtain

$$\overline{V}_3 = \frac{f}{\eta}\left(\frac{\overline{g}}{\overline{g''}}\right)\left(\frac{\partial\overline{p}}{\partial x_3}\right) = \frac{bf}{2\eta\alpha}\left[\frac{\sinh\alpha b}{1 - \cosh\alpha b} + \frac{2}{\alpha b}\right]\left(\frac{\partial\overline{p}}{\partial x_3}\right).$$

Therefore, Darcy's law is

$$\overline{V}_3 = \frac{bf^2}{2\eta\alpha}\left[\frac{\sinh\alpha b}{1 - \cosh\alpha b} + \frac{2}{\alpha b}\right](\mathrm{grad}\,\overline{p})_3.$$

If we substitute into the above formula the power series expressions for the hyperbolic sine and hyperbolic cosine, we get after some manipulation

$$\overline{V}_3 = -\frac{b^2 f^2}{\eta}\left[\frac{\left(\frac{1}{3!}-\frac{2}{4!}\right)+\left(\frac{1}{5!}-\frac{2}{6!}\right)\alpha^2 b^2+\cdots}{1+2\frac{\alpha^2 b^2}{4!}+2\frac{\alpha^4 b^4}{6!}+\cdots}\right](\mathrm{grad}\,\overline{p})_3, \tag{29}$$

where $\alpha^2 = \frac{2}{3} f^3 \left(\frac{1}{f^3}\right)''$.

The important facts about the above formula are as follows:

(i) It is a generalization of Darcy's law for the flat, Euclidean space. Darcy's law in this particular case is obtained by setting $f = 1$ (and $\alpha = 0$) in (29).

(ii) It is defined for arbitrary conformal factors f, even for the ones for which the corresponding symmetric unidirectional flows do not exist!

In the case of a hyperbolic 3-space of curvature $-K$, we have

$$\alpha^2 = 4Kf^2 = \frac{4K}{(1+\sqrt{K}x_3)^2}.$$

Since $x_3 > 0$, this expression is less than or equal to $4K$, and thus a near-zero choice of curvature will make α^2 uniformly small.

5. Summary and concluding remarks

In this paper, we have considered the generalization of fluid flow to non-Euclidean spaces by obtaining Stokes equation for symmetric unidirectional flows in a smooth orientable manifold of dimension 3. We have also found the conditions under which Stokes equation is separable. As examples we recovered Stokes equation in $\mathbb{R}^3$ with the Euclidean metric both in cylindrical and spherical coordinates. This was also done for $\mathbb{R}^3$ with Minkowski metric. We studied then the case of a flow in $\mathbb{R}^3$ with a conformal metric and found that a restriction on the conformal factor is needed in order to have non-trivial solutions of Stokes equation. This restriction rules out spaces of constant curvature such as the hyperbolic 3-space. The existence of symmetric unidirectional flows was addressed and a condition on the manifold metric established for Stokes equation to have solutions. These conditions rules out symmetric unidirectional flows in conical and toroidal geometries. In the conical case, the problem seems to come from the curvature singularity at the cone vertex. In [20], this problem was avoided by cutting out the vertex in order to provide the inlet for the flow. Darcy's law was finally obtained for the cases of two pseudo-spheres in Minkowiski space and for two parallel planes in $\mathbb{R}^3$ with conformal metric. The latter case considers even the case where symmetric unidirectional flows are not possible. A series expansion of Darcy's law for small values of the parameter of separation of the two parallel planes recovered Darcy's law in Euclidean space in the unit conformal factor limit.

Although real Laplacian growth processes including viscous fingering, diffusion-limited aggregation [25] and dendritic solidification [26] sometimes occur on curved surfaces, such as cell membranes or porous rock formations, the theoretical studies on these models mostly assume a flat Euclidean surface. Overall, the influence of a curved substrate has been largely neglected in the literature. In this sense, the results we present in this work add a welcome versatility to the traditional viscous fingering problem. Our purpose is not only to study growth on a particular background shape, but more generally to be able to explore the effects of the local surface geometry on pattern formation. The rigorous derivation of Darcy's law and the establishment of the conditions for symmetric flows in curved spaces are basic initial requirements for the development of systematic ways of controlling fingering instabilities by geometric means. So, this study is the starting point for the investigation of key nonlinear aspects of the Saffman–Taylor problem such as finger competition and finger tip splitting in a variety of curved Hele–Shaw geometries. It is hoped our work will motivate other theoretical and experimental groups to examine how geometric and topological features may impose restrictions on the shapes of the patterned structures formed in viscous fingering and other physical systems.

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